UDC 539.3

ON THE GAME THEORY APPROACH TO PROBLEMS OF OPTIMIZATION OF ELASTIC BODIES

PMM Vol. 37, №6, 1973, pp. 1098-1108
N. V. BANICHUK
(Moscow)
(Received April 3, 1973)

Problems of optimization of elastic bodies are considered usually in deterministic formulation, and for their solution the methods of variational calculus and the theory of optimal control are applicable (c. f., e. g., [1] and [2 – 4]). In the present paper there are considered those cases when either the complete information concerning the applied loads is not available, or it is known that the structure may be subjected subsequently to various loads of a certain class. The formulation is given of the problem of the determination of the shape of the elastic body, optimal for a class of loads, and there is indicated a general scheme for its solution based on the "minimax" approach used in the theory of games. Problems of optimization of elastic beams are considered and as a result of their solution certain features of optimal shapes are exhibited.

1. Formulation of the problem. We consider the problem of determination of the optimal form of an elastic body in state of equilibrium under the action of applied forces. We write the equilibrium equations in the form

$$L(v)\,u = f \tag{1.1}$$

Here $u = (u_1(x), \ldots, u_n(x))$ is a vector-function which defines the state of the elastic medium; $f(x)$ is the vector of the external actions; x is the vector of the space coordinates, taking values in some given domain D, occupied by the elastic body. In concrete problems, the components of the vector u can be chosen to be the stress-tensor components σ_{ij}, the strain-tensor components ε_{ij}, the displacements w_i, the moments M_i which occur in the deformable medium, and so on. In (1.1), by $L(v)$ we have denoted a differential operator with respect to the space coordinates x_i. The coefficients of the operator depend on the vector-function $v = (v_1(x), \ldots, v_m(x))$. The functions $v_i(x)$ determine the form of the deformable body and, in the problem under consideration, they play the role of controls. As functions $v_i(x)$, there may appear the distribution of the thickness and the areas of cross sections of the body and also functions which serve for the description of the geometry of the construction (for example, functions which define the position of the line joining the centers of the cross sections of a curvilinear beam).

The requirements claimed for structures lead to constraints on the controls

$$v \in V \tag{1.2}$$

By V we have denoted some given set of admissible controls. If, for example, for the control v we consider the distribution of the thickness of plates or beams, then the constraint (1.2) may have the form $\delta_1 \leqslant v(x) \leqslant \delta_2$, where δ_1, δ_2 are specified positive numbers ($\delta_1 \leqslant \delta_2$).

The form of the load applied to the body is not fixed in advance but we assume that a set F which contains all possible realizations of the exterior forces, is specified. We write this in the form

$$f \in F \tag{1.3}$$

and in order to solve optimization problems for the form of the body, we will take into consideration only forces from (1.3). If, for example, the optimization object is a plate of variable thickness and the external actions are one-sided forces resultant of which do not exceed the value of P, then the set F in (1.3) has the form $F\ (f > 0,\ \int f(x) \times dx \leqslant P,\ x \in D)$.

External actions can occur also in the boundary conditions for Eq. (1.1). The form of the boundary conditions depends essentially on the problem to be solved and here in these general considerations it will not be concretely defined.

The optimization problem consists in finding of the function $v(x)$ from (1.2) which minimizes the functional (the weight of the body)

$$G(v) = \gamma \int_D \varphi(v)\, dx \to \min \tag{1.4}$$

and satisfies for any f from (1.3) the strength and geometric conditions

$$\omega(x, y, u, u_x, v, v_x) \leqslant 0 \tag{1.5}$$

By γ we have denoted the specific weight of the material of the medium, while the quantity φ, which occurs in (1.4), is a specified function of v. The value $\omega = (\omega_1, \ldots, \omega_r)$ is a specified vector-function of its arguments and the condition (1.5) represents a system of inequalities. The vector variable $y \in Y$ denotes that part of the space coordinates which do not occur in the expression of the operator L and in the formula (1.4).

For example, if we consider the problem of the bending of a plate in the plane x_1x_2, then the variable y can be taken to be the coordinate which gives the distance along the normal from the neutral (median) surface of the plate. The domain Y of the variation of this coordinate is determined by the position of the plate surface. In concrete problems we may have as inequalities (1.5) restrictions of different types. One type are the strength conditions, reducing to constraints on the stresses. As strength conditions we can consider, for example, the conditions $|\sigma_{ij}| \leqslant \sigma_{ij}^\circ$ (where σ_{ij}° are specified positive constants) which limit separately the allowable values of each component of the stress tensor, or the condition $g(\sigma_{ij}) - k^2 \leqslant 0$ (where k is the plasticity constant) representing the criterion of translation of the medium into the plastic state. To another type belong the limits imposed on elastic displacements, arising from the geometric or stiffness requirements claimed for the structure. As an example we present the conditions $|w_i| \leqslant e_i$ (where e_i are specified positive constants) which limit the allowable deflections of the deformed body. Possible are also combined limits on stresses and displacements. An analysis of various restrictions considered in structural optimization and a review of the results obtained is contained in [2 – 4].

The formulated problem (1.1) – (1.5), because of the existing indeterminacy in the specific form of the applied loading, belongs to game problems (natural game) and for its solution we can apply the minimax (or guaranteed) approach.

2. The minimax approach. We indicate a method which allows us, in a series of cases, to reduce the solution of the problem (1.1) – (1.5) to the determination of the extrema of some variational problem. We assume that the solution of the boundary value problem for Eq. (1.1) with the corresponding boundary conditions and with the constraints (1.2), (1.3) can be found in a closed form $u = u(x, f, v, v_x)$. The dependence of u on f and v can be, in general, a functional one. We substitute the expression for u into the left-hand sides of the inequalities (1.5). As a result, we arrive at a system of functional inequalities

$$\Omega_j (x, y, f, v, v_x) \leqslant 0 \qquad (j = 1, \ldots, r) \tag{2.1}$$
$$\Omega_j (x, y, f, v, v_x) \equiv \omega_j (x, y, u(x, f, v, v_x), u_x(x, f, v, v_x), v, v_x)$$

We determine the maximum of the quantity $\Omega_j (x, y, f, v, v_x)$ with respect to the variables $f \in F$ and $y \in Y$. The maximization is carried out for fixed $x \in D$ and $v \in V$. We assume that the maximum of the jth component of the vector Ω, i.e. of the quantity Ω_j, is attained for $f_j = f_j^*$ and $y_j = y_j^*$, i.e.

$$\Omega_j (x, y_j^*, f_j^*, v, v_x) = \max_{y \in Y} \max_{f \in F} \Omega_j (x, y, f, v, v_x)$$

and we introduce the notation

$$\Omega_j^* (x, v, v_x) \equiv \Omega_j (x, y_j^* (x, v, v_x), \; f_j^* (x, v, v_x), \; v, \; v_x) \qquad (j = 1, \ldots, r) \tag{2.2}$$

If the maximum of Ω_j with respect to f is attained at once at several different functions from (1.3), then for f_j^* we can take any of these functions. We can proceed exactly in the same way if the quantity y which realizes the maximum of the function Ω_j is not unique. Making then use of the conditions (2.1) and the notation (2.2), we arrive at the inequalities

$$\Omega_j^*(x, v, v_x) \leqslant 0 \qquad (j = 1, \ldots, r) \tag{2.3}$$

Thus, the initial problem (1.1) – (1.5) reduces to the variational problem of minimizing with respect to v the integral (1.4) under the differential inequalities (2.3) and the conditions (1.2) imposed on the vecter-function v In order to solve this problem we can make use of methods developed in the theory of optimal control.

In solving problems on the basis of the indicated approach, one of the two possibilities can occur. Either it turns out that in the class under consideration there exists a "worst" load for which the structure of minimum weight, found in the analysis for this load only, satisfies the conditions of strength and stiffness (1.5), also for all other applied loads from the given class (a structure of this shape is precisely optimal for this class of forces, i.e. it is the solution to the original problem). Or the worst load does not exist and the solution optimal for the class of loads is not optimal for any single load of the given set. Examples of both types will be presented below.

Let us note that the minimax approach may be applied as well to problems with incomplete information concerning boundary conditions and properties of materials from which the structure is fabricated.

The procedure described above of reducing problems of game theory to the problem of calculus of variations is applicable in those cases when it is possible to obtain the dependence of elastic solutions on controls in explicit form. This limits the applicability of the method in solving analytically multidimensional problems of structural optimization.

3. Optimization for the class of loads of elastic beams. The equilibrium of an elastic beam of length l, situated in the plane xy and loaded by the external forces $f(x)$, parallel with the y-axis, is described by the equations

$$dM/dx = Q, \quad dQ/dx = -f \tag{3.1}$$

Here $M = M(x)$ and $Q = Q(x)$ are, respectively, the bending moment and the transverse force, acting in a cross section of the beam, perpendicular to the x-axis. In the undeformed state the beam is situated along the x-axis and is fixed either at both endpoints $x = 0$ and $x = l$, or at the endpoint $x = 0$. The beam has a rectangular cross section of constant width $v_2 = a$ and variable height $v_1 = v_1(x)$. The function $v_1 = v_1(x)$, which defines the form of the beam, is the desired quantity. We assume that the load applied to the beam is positive (the direction of the action of the load coincides with the positive direction of the y-axis) and its resultant does not exceed a given quantity P, i.e.

$$f(x) \geqslant 0, \qquad \int_0^l f(x)\,dx \leqslant P \tag{3.2}$$

For any realization of the load, satisfying condition (3.2), the normal and the shear stresses σ_x and τ_{xy} must satisfy the strength conditions of the beam

$$\omega_1 \equiv |\sigma_x| - \sigma_0 \leqslant 0, \quad \omega_2 \equiv |\tau_{xy}| - \tau_0 \leqslant 0 \tag{3.3}$$

where σ_0, τ_0 are specified constants, and are computed from the formulas

$$\sigma_x = \frac{My}{J}, \qquad \tau_{xy} = \frac{1}{a}\frac{\partial}{\partial x}\left(\frac{Md}{J}\right) \tag{3.4}$$
$$J = \frac{av_1^3}{12}, \quad d = \frac{a}{2}\left(\frac{v_1^2}{4} - y^2\right), \quad -\frac{v_1}{2} \leqslant y \leqslant \frac{v_1}{2}$$

Here and below, the coordinate y is measured from the center of the cross section of the beam and varies between the indicated limits.

The problem of the optimization of the form of the beam consists in finding of the function $v_1 = v_1(x)$, satisfying the conditions (3.3) (where σ_x and τ_{xy} are calculated according to (3.1), (3.4)) for any realizations $f = f(x)$ from (3.2), and minimizing the integral (the weight of the beam)

$$G = a\gamma \int_0^l v_1(x)\,dx \tag{3.5}$$

4. The determination of the optimal form of articulated beam. The boundary conditions for the equations (3.1) in the case of a beam with articulated endpoints have the form

$$M(0) = M(l) = 0 \tag{4.1}$$

Before we proceed to the solution of the problem, we clarify some properties of the functions $M(x)$ and $Q(x)$, needed in the sequel. To this end, we integrate Eqs. (3.1) with the indicated boundary conditions. As a result we obtain

$$M(x) = \int_0^l K(x,t)\,f(t)\,dt, \qquad Q(x) = \int_0^l T(x,t)\,f(t)\,dt \tag{4.2}$$

$$K(x,t)=\frac{t(l-x)}{l}\quad\text{for}\quad 0\leqslant t\leqslant x,$$

$$K(x,t)=\frac{x(l-t)}{l}\quad\text{for}\quad x\leqslant t\leqslant l$$

$$T(x,t)=-\frac{t}{l}\quad\text{for}\quad 0\leqslant t\leqslant x,$$

$$T(x,t)=1-\frac{t}{l}\quad\text{for}\quad x<t\leqslant l$$

From the positivity of the functions $K(x, t)$ and $f(t)$ there follows that $M(x)\geqslant 0$. We fix the point $x\in[l/2, l]$ and we consider the set of values taken by the quantities $M(x)$ and $Q(x)$ for all possible realizations $f=f(t)$ from (3.2). We denote by $\max_f M(x)$ and $\max_f Q(x)$ the maximum values of the bending moment and of the modulus of the transverse force and we will prove that for $l/2\leqslant x\leqslant l$ we have

$$\max_f M(x)=Px\left(1-\frac{x}{l}\right),\qquad \max_f|Q(x)|=\frac{Px}{l}\tag{4.3}$$

To this end, making use of the formulas (4.2), we perform the following estimates:

$$M(x)\leqslant\max_t K(x,t)\int_0^l f(t)\,dt=Px\left(1-\frac{x}{l}\right)\tag{4.4}$$

The maximum with respect to t in (4.4) is computed for $0\leqslant t\leqslant l$. We also note that for the realization $f(t)=P\delta(t-x)$ the moment $M(x)=Px(l-x)/l$. By δ we have denoted the δ-function. From here and from the above given estimate (4.4) we obtain the validity of the formula (4.3) for M

The proof of the relation (4.3) for Q is carried out with the aid of the similar estimates

$$|Q(x)|\leqslant\text{vrai}\max_t|T(x,t)|\int_0^l f(t)\,dt=xP/l\tag{4.5}$$

Here by $\text{vraimax}_t\ |T|$ we have denoted the essential maximum with respect to t $(0\leqslant t\leqslant l)$ of the piecewise-continuous function $T(x, t)$ which is discontinuous at $t=x$. We substitute the realization $f(t)=P\delta(t-x^1)$ with $l/2\leqslant x^1<x$ into the formula (4.2) for Q and we compute the integral. As a result we obtain $Q(x)=Px^1/l$. At the limit for $x^1\to x-0$ we have $\lim|Q(x)|=Px/l$. From here, taking into account the inequality (4.5), we obtain the formula (4.3) for Q.

Taking into account the symmetry of the conditions of the problem with respect to the point $x=l/2$, in what follows we will carry out all arguments only for $l/2\leqslant x\leqslant l$.

Using the above mentioned properties of the functions $M(x)$ and $Q(x)$, we proceed to the finding of explicit expressions for $\Omega_1{}^*$ and $\Omega_2{}^*$. First we determine the quantities Ω_1 and Ω_2. Making use of the formulas (2.1), (3.4), (3.7), (4.2), we obtain

$$\Omega_1=\frac{12y}{av_1^3}\int_0^l K(x,t)f(t)\,dt-\sigma_0\tag{4.6}$$

$$\Omega_2=\frac{6}{a}\left|\left(\frac{1}{4v_1}-\frac{y^2}{v_1^3}\right)\int_0^l T(x,t)f(t)\,dt+\left(\frac{3y^2}{v_1^4}-\frac{1}{4v_1^2}\right)\frac{dv_1}{dx}\int_0^l K(x,t)f(t)\,dt\right|-\tau_0$$

The expression for the function $\Omega_1{}^*$, obtained after use of formulas (2.2), (4.3), (4.6) and after computing the maxima of the function Ω_1 with respect to f from (3.2) and with respect to y from the interval $-v_1/2 \leqslant y \leqslant v_1/2$, has the form

$$\Omega_1{}^* = \frac{6Px}{av_1^2}\left(1-\frac{x}{l}\right) - \sigma_0 \tag{4.7}$$

Let us determine the function $\Omega_2{}^*$. To this end we use the formulas (2.2), (4.3), (4.6). The expression which occurs between the absolute value signs in the formula (4.6) for Ω_2 is a linear function relative to y^2 and, consequently, the maximum with respect to y of the function Ω_2 for $-v_1/2 \leqslant y \leqslant v_1/2$ is attained either for $y^2 = v_1^2/4$ or for $y^2 = 0$. Computing the indicated maxima, we write the expression for $\Omega_2{}^*$ in the form

$$\Omega_2{}^* = \max(\Psi_1, \Psi_2) - \tau_0 \tag{4.8}$$

$$\Psi_1 = \max_f \frac{3}{2av_1}\left|\int_0^l T(x,t)f(t)\,dt - \frac{1}{v_1}\frac{dv_1}{dx}\int_0^l K(x,t)f(t)\,dt\right|$$

$$\Psi_2 = \max_f \frac{3}{av_1^2}\left|\frac{dv_1}{dx}\int_0^l K(x,t)f(t)\,dt\right|$$

Applying formula (4.3) for the computation of the function Ψ_2, we find

$$\Psi_2 = \frac{3Px}{av_1^2}\left(1-\frac{x}{l}\right)\left|\frac{dv_1}{dx}\right| \tag{4.9}$$

Proceeding in the same way as in the proof of the relation (4.3) for Q, we obtain

$$\Psi_1 = \max\left(\frac{3P}{2av_1}\left|\frac{x}{l} + \frac{x}{v_1}\left(1-\frac{x}{l}\right)\frac{dv_1}{dx}\right|, \quad \frac{3P}{2av_1}\left|\frac{x}{l} - 1 + \frac{x}{v_1}\left(1-\frac{x}{l}\right)\frac{dv_1}{dx}\right|\right) \tag{4.10}$$

Making use of the obtained expressions for the functions $\Omega_1{}^*$ and $\Omega_2{}^*$, we give the conditions which must be satisfied by the function $v_1 = v_1(x)$ in order that the inequalities (2.3) should hold. The first of the inequalities (2.3), after substitution in it the expression (4.7) for $\Omega_1{}^*$, leads to the condition

$$v_1(x) \geqslant W_1(x) \equiv \left[\frac{6Px}{a\sigma_0}\left(1-\frac{x}{l}\right)\right]^{1/2} \tag{4.11}$$

Substituting the expression for $\Omega_2{}^*$ from (4.8) – (4.10) into the second inequality of (2.3), we have ($\lambda = al\tau_0/3P$)

$$\frac{dv_1}{dx} \leqslant \frac{1}{x(l-x)}\min(\lambda v_1^2, 2\lambda v_1^2 - xv_1, 2\lambda v_1^2 + (l-x)v_1) \tag{4.12}$$

$$\frac{dv_1}{dx} \geqslant \frac{1}{x(l-x)}\max(-\lambda v_1^2, -2\lambda v_1^2 - xv_1, -2\lambda v_1^2 + (l-x)v_1) \tag{4.13}$$

The inequalities (4.12), (4.13) can be simplified if we note that the third expression written between the parantheses in (4.12) is greater than the second one and that in (4.13) the second expression is smaller than the third one. Taking this into account, we write the inequalities (4.12), (4.13) in the following manner:

$$\frac{dv_1}{dx} \leqslant \frac{1}{x(l-x)} \min(\lambda v_1^2, 2\lambda v_1^2 - xv_1) \tag{4.14}$$

$$\frac{dv_1}{dx} \geqslant \frac{1}{x(l-x)} \max(-\lambda v_1^2, -2\lambda v_1^2 + (l-x)v_1) \tag{4.15}$$

We divide the domain $S\ (l/2 \leqslant x \leqslant l,\ v_1 \geqslant 0)$, in which we look for the solution of the optimal problem, into three subdomains

$$S_1\ (l/2 \leqslant x \leqslant l,\quad v_1 \geqslant x/\lambda);\quad S_2\ (l/2 \leqslant x \leqslant l,\quad (l-x)/\lambda \leqslant v_1 \leqslant x/\lambda)$$
$$S_3\ (l/2 \leqslant x \leqslant l,\ 0 \leqslant v_1 \leqslant (l-x)/\lambda)$$

In the subdomains indicated the inequalities (4.14), (4.15) can be written in the form

$$-\frac{\lambda v_1^2}{x(l-x)} \leqslant \frac{dv_1}{dx} \leqslant \frac{\lambda v_1^2}{x(l-x)}, \quad (x, v_1) \in S_1 \tag{4.16}$$

$$-\frac{\lambda v_1^2}{x(l-x)} \leqslant \frac{dv_1}{dx} \leqslant \frac{2\lambda v_1^2 - xv_1}{x(l-x)}, \quad (x, v_1) \in S_2 \tag{4.17}$$

$$-\frac{2\lambda v_1^2 - (l-x)v_1}{x(l-x)} \leqslant \frac{dv_1}{dx} \leqslant \frac{2\lambda v_1^2 - xv_1}{x(l-x)}, \quad (x, v_1) \in S_3 \tag{4.18}$$

The inequalities (4.16) are consistent for any $(x, v_1) \in S_1$. For the solvability of the inequalities (4.17) in the domain S_2 and of the inequalities (4.18) in the domain S_3 the following conditions

$$v_1 \geqslant x/(3\lambda), \quad v_1 \geqslant l/(4\lambda) \tag{4.19}$$

must be satisfied.

Thus, the initial problem of the optimization of articulated rectangular beams of variable height has been reduced to the determination of a continuous function $v_1(x)$, satisfying the finite inequalities (4.11), (4.19), the differential inequalities (4.16) – (4.18) and minimizing the integral

$$G = 2a\gamma \int_{l/2}^{l} v_1(x)\,dx \to \min \tag{4.20}$$

The functions $v_1 = v_1(x)$, satisfying the inequalities (4.11), (4.16) – (4.19), will be called admissible and here we investigate some of their properties which will be used in the sequel. The admissible functions must satisfy the differential inequality

$$\frac{dv_1}{dx} \leqslant \frac{2\lambda v_1^2 - xv_1}{x(l-x)} \equiv g_1(x, v_1)$$

appearing in (4.17) and (4.18). Let us consider an arbitrary admissible function $v_1(x)$, satisfying the condition $v_1(x^\circ) = v_1^\circ$ and representing the solution of some differential equation $dv_1/dx = g_2(x, v_1)$, whose right-hand side, by virtue of the above given inequality, is estimated in the following manner: $g_2(x, v_1) \leqslant g_1(x, v_1)$. Together with the equation for v_1 we examine the equation $dh/dx = g_1(x, h)$ with the initial condition $h(x^\circ) = v_1^\circ$. From the theorem on the comparison of the solutions of differential equations we obtain the estimate $v_1(x) \leqslant h(x)$ for $x^\circ \leqslant x < l$.

The expression for $h(x)$, obtained as a result of the integration of the equation $dh/dx = g_1$ and the determination of the arbitrary constant from the condition $h(x^\circ) = v_1^\circ$, has the form

$$h = \frac{l-x}{2\lambda \ln(\mu/x)}, \qquad \mu = x^\circ e^{(l-x^\circ)/2\lambda v_1^\circ} \tag{4.21}$$

Let us investigate the behavior of the integral curves $h(x)$ in dependence of the position of the initial point (x°, v_1°). If the values x°, v_1° are such that the constant $\mu > l$, then, as we can easily see from (4.21), $h(x) \to 0$ for $x \to 0$. In this case, taking into account the above proved inequality $v_1(x) \leqslant h(x)$, for the admissible function $v_1(x)$ we have $v_1(x) \to 0$ for $x \to l$.

For initial values x°, v_1° such that $\mu < l$, the integral curves $h(x)$ diverge at infinity for x tending to μ. If however the quantities x°, v_1° satisfy the condition

$$x^\circ \frac{(l - x^\circ)}{\exp 2\lambda v_1^\circ} = l$$

then for $x \to l$ there is an indeterminacy in the formula (4.21) for h, which, removed by l'Hospital's rule, gives $h(l) = l / 2\lambda$. The function $h(x)$ from (4.21) with $\mu = l$ will be denoted by $W_2(x)$.

Making use of these properties of the function $h(x)$, we show that the solution of the optimal problem (4.11), (4.16) – (4.20) has the form

$$v_1 = \begin{cases} W_1(x), & l/2 \leqslant x \leqslant x^* \\ W_2(x), & x^* \leqslant x \leqslant l \end{cases} \tag{4.22}$$

$$W_1(x) \equiv \left[\frac{6Px}{a\sigma_0}\left(1 - \frac{x}{l}\right)\right]^{1/2}, \qquad W_2(x) \equiv \frac{l - x}{2\lambda \ln(l/x)}$$

The quantity x^* is determined in the following manner. Let ξ be a root of the equation $W_1(\xi) = W_2(\xi)$, which can be written in the form

$$\left[\frac{\xi}{\varkappa(l - \xi)}\right]^{1/2} \ln\left(\frac{\xi}{l}\right) = \frac{1}{4}, \qquad \varkappa = \frac{6P\sigma_0}{al\tau_0^2} \tag{4.23}$$

If ξ satisfies the inequality $l / 2 \leqslant \xi < l$, then we set $x^* = \xi$. Otherwise, the quantity x^* in the formulas (4.22) is taken to be equal to $x^* = l / 2$. The first case, as can be easily verified, occurs for $\varkappa \leqslant 16 (\ln 2)^2$ and the second one for $\varkappa \geqslant 16 (\ln 2)^2$.

The proof of the optimality of the function (4.22) is concluded in the verification of the fact that the given function is admissible, i.e. satisfies the conditions (4.11), (4.16) – (4.19), and that there is no other admissible function $v_1(x)$ for which the functional (4.20) takes a smaller value than for v_1 given by (4.22).

First we consider the case $\varkappa \leqslant 16 (\ln 2)^2$ and we determine the intervals of the variation of x, for which the inequalities (4.16) – (4.18) hold by the substitution $v_1 = W_1(x)$. Performing elementary computations, we obtain that the inequality (4.16) is satisfied for $l / 2 \leqslant x \leqslant \alpha \equiv {}^1/_2 l\,(1 + \sqrt{4/(4+\varkappa)})$, the inequality (4.18) for $l/2 \leqslant x \leqslant c \equiv {}^1/_8 l\,(4 + \sqrt{16 - \varkappa})$, and the inequality (4.17) for $l / 2 \leqslant x \leqslant \beta \equiv \min(\alpha, c)$. We can verify that $x / 2\lambda < W_2(x) < x / \lambda$ for $l / 2 \leqslant x < l$, $W_2(x) = x / 2\lambda$ for $x = l$. From here, in particular, it follows that the quantity $x^1 = 16l / (\varkappa + 16)$, the root of the equation $x^1 / 2\lambda = W_1(x^1)$ satisfies the inequality $x^1 \geqslant x^*$. Therefore, for the proof of the inequalities $\alpha \geqslant x^*$, $\beta \geqslant x^*$, $c \geqslant x^*$ it is sufficient to show that $\alpha \geqslant x^1$, $\beta \geqslant x^1$, $c \geqslant x^1$ for $0 \leqslant \varkappa \leqslant 16 (\ln 2)^2$, which is obtained by elementary computations. The curve $W_2(x)$ from (4.22) is either in the domain S_2 or in $S_2 + S_3$ ($W_2(x) \leqslant x / \lambda$), depending on the value of the parameter $\varkappa$. The inequalities (4.17), (4.18) will be satisfied since for $v_1 = W_2(x)$ in the right-hand sides of these inequalities the equality signs prevail. This follows from the construction of the function $W_2(x)$. Consequently, the function $v_1(x)$, specified by the equalities (4.22), is admissible for $0 \leqslant \varkappa \leqslant 16.(\ln 2)^2$. For $\varkappa \geqslant 16(\ln 2)^2$ the function $v_1 = W_2(x)$ $(l/2 \leqslant x \leqslant l)$ is also admissible, since for $v_1 = W_2(x)$ the conditions (4.17), (4.18) will be satisfied.

Let us show now that the admissible function $v_1(x)$ from (4.22) is optimal. For this it is sufficient to prove that the graph of any other admissible function $v_1(x)$ is situated not below the curves $W_1(x)$ and $W_2(x)$. For $l/2 \leqslant x \leqslant x^*$, the admissible $v_1(x)$, as it follows from (4.12), must be situated not below the curve $W_1(x)$, i.e. $v_1(x) \geqslant W_1(x)$. Let us prove that for $x^* \leqslant x \leqslant l$ the inequality $v_1(x) \geqslant W_2(x)$ is satisfied. Let us assume the opposite, i.e. that at some point $x = x^1$ $(x^* \leqslant x^1 \leqslant l)$ the admissible function $v_1(x)$ satisfies the inequality $v_1(x^1) < W_2(x^1)$. But, as mentioned above, for the admissible curve $v_1(x)$ which passes through the point $(x^1, v_1(x^1))$, we have $v_1(x) \to 0$ for $x \to l$. Therefore the trajectory $v_1(x)$ originating from the point $(x^1, v_1(x^1))$, inevitably hits the prohibited domain defined by the inequalities (4.19) and thus the assumption of the admissibility of the function $v_1(x)$ will be violated. The obtained contradiction proves the validity of inequality $v_1(x) \geqslant W_2(x)$ for $x^* \leqslant x \leqslant l$. Consequently, the function $v_1(x)$ defined by the formula (4.22), is optimal.

For the obtained optimal solution we estimate the magnitudes of the stresses σ_x and τ_{xy} which appear when the concentrated force P acts on the beam. We denote here by σ_x the maximum value of the normal stress along the transversal section (i.e. with respect to y), and by τ_{xy} the essential maximum of the shear stresses. If the concentrated force P is applied to the optimal beam at the point ξ $(l/2 \leqslant \xi < x^*)$, then the quantities $\sigma_x(x)$ and $\tau_{xy}(x)$, as it follows from the formulas (3.4), (4.2), (4.22), satisfy the inequalities $\sigma_x(x) \leqslant \sigma_0$ and $\tau_{xy}(x) < \tau_0$ $(l/2 \leqslant x \leqslant l)$. The equality $\sigma_x = \sigma_0$ is attained for $x = \xi$. If however $x^* \leqslant \xi \leqslant l$, then $\sigma_x \leqslant \sigma_0$, $\tau_{xy} \leqslant \tau_0$. In this case the limiting value of the shear stress $\tau_{xy} = \tau_0$ is attained for $x = \xi$, while the equality $\sigma_x = \sigma_0$ holds if $x = \xi = x^*$.

Thus, by applying a concentrated force to the optimal beam at any point x from the interval $(l/2, l)$, the limiting state is attained only at this point and, consequently, at the designing of a beam with a fixed load there will be additional possibilities for optimization. This immediately confirms the determination of the optimal form for fixed forces (the corresponding computations are not given here). Consequently, for the problem under consideration there is no worst (in the above indicated sense) load and the optimal form of the beam for the class of the forces is not optimal for any of the individually realization of the force from the given class.

5. The optimal form of a cantilever beam. For cantilever beams which are fixed at the point $x = 0$, the distributions of the transverse forces $Q(x)$ and of the bending moments $M(x)$ have the form

$$Q(x) = \int_x^l f(t)\,dt, \qquad M(x) = \int_x^l (x-t) f(t)\,dt \tag{5.1}$$

$$\max_f Q(x) = P, \quad \max_f |M(x)| = P(l-x) \tag{5.2}$$

The maximum of $Q(x)$ is realized for any fixed x from the interval $0 < x < l$, when all the load is applied at the right-hand side of the point x ($f(t) = 0$ for $t < x$).

The load which furnishes the maximum for $|M(x)|$, is a concentrated force of magnitude P, applied at the free end of the beam at the point $x = l$, i.e. $f(t) = P\delta(t - l)$. The formulated assertions can be proved in the same way as it has been done in Sect. 4 in the investigation of the corresponding properties of the quantities Q and M.

For the determination of the functions Ω_1^* and Ω_2^* we make use of the formulas (2.2), (3.3), (3.4), (5.1), (5.2) and, without carrying out the explicit computations which are also similar to the corresponding computations of Sect. 4, we give the final formulas as

$$\Omega_1^* = \frac{6P(l-x)}{av_1^2} - \sigma_0 \tag{5.3}$$

$$\Omega_2^* = \begin{cases} \max\left[\frac{3P}{2av_1}\left(1 - \frac{(l-x)}{v_1}\frac{dv_1}{dx}\right), \frac{3P(l-x)}{av_1^2}\frac{dv_1}{dx}\right] - \tau_0, & \frac{dv_1}{dx} > 0 \\ \max\left[\frac{3P}{2av_1}, \frac{3P(x-l)}{av_1^2}\frac{dv_1}{dx}\right] - \tau_0, & \frac{dv_1}{dx} < 0 \end{cases} \tag{5.4}$$

Substituting the expressions (5.3), (5.4) into the conditions (2.3), we obtain that for the fulfilment of these conditions it is necessary that the function $v_1(x)$ satisfies the inequalities

$$v_1(x) \geqslant \left[\frac{6P(l-x)}{a\sigma_0}\right]^{1/2}, \qquad v_1(x) \geqslant \frac{3P}{2a\tau_0}$$

We consider now the continuous function

$$v_1 = \begin{cases} \left[\frac{6P(l-x)}{a\sigma_0}\right]^{1/2}, & 0 \leqslant x \leqslant x^* \\ \frac{3P}{2a\tau_0}, & x^* \leqslant x \leqslant l \end{cases} \tag{5.5}$$

$$(x^* = l - 3P\sigma_0 / 8a\tau_0^2)$$

and we prove that it describes the desired optimal form, i.e. $v_1 = v_1(x)$ supplies the minimum of the functional (3.5), considered in the class of the continuous functions $v_1 = v_1(x)$, satisfying the inequalities (2.3) with Ω_1^* and Ω_2^* from (5.3), (5.4). Since the conditions (5.4) are necessary, and other admissible (in the sense of these conditions) function $v_1(x)$ will lie not below the graph of the function (5.5), and, consequently, it will impart to the functional (3.5) the larger value. It can be also easily verified, carrying out the computation with the formulas (5.3) – (5.5), that $\Omega_1^* \leqslant 0$ and $\Omega_2^* \leqslant 0$ for $0 \leqslant x \leqslant l$. Consequently, v_1 from (5.5) gives the optimal form.

We mention some properties of the obtained optimal solution. We denote here by σ_x and τ_{xy} the maximal values of these quantities across the transverse section of the beam (i.e. with respect to the variable y). If the concentrated force P is applied at the free end of the beam ($\xi = l$), then for $0 \leqslant x \leqslant x^*$, according to the formulas (3.4), (5.1), (5.5) we have $\sigma_x(x) = \sigma_0$, $\tau_{xy}(x) \leqslant \tau_0$ ($\tau_{xy} = \tau_0$ at the point $x = x^*$, while on the segment $x^* < x \leqslant l$ we have $\tau_{xy}(x) = \tau_0$, $\sigma_x(x) < \sigma_0$. If however the force P is applied at the point $\xi \in (x^* \leqslant \xi < l)$, then for $0 \leqslant x \leqslant \xi$ we have $\sigma_x(x) < \sigma_0$, $\tau_{xy}(x) \leqslant \tau_0$ ($\tau_{xy}(x) = \tau_0$, $x^* \leqslant x \leqslant \xi$), while for $\xi < x \leqslant l$ we have the equalities $\sigma_x(x) = \tau_{xy}(x) = 0$. In the case when $0 < \xi < x^*$, then on the entire segment $0 \leqslant \xi \leqslant l$ the inequalities $\sigma_x < \sigma_0$, $\tau_{xy} < \tau_0$ are satisfied.

Thus, the simultaneously limiting states are attained in all cross sections of the beam only under the action of a concentrated force applied at the free end of the beam. Therefore, for the problem under consideration (in contrast to the problem discussed in Sect. 4), there exists the worst force in the class F, for which the optimal solution obtained with regard solely to that force, is also optimal for the entire class as a whole.

The author thanks F. L. Chernous'ko for giving consideration to this paper.

REFERENCES

1. Lur'e, A. I., The application of the maximum principle to the simplest problems in mechanics. Tr. Leningr. Politekhn. Inst., № 252, Leningrad-Moscow, "Mashinostroenie", 1965.
2. Wasiutynski, Z. and Brandt, A., The present state of knowledge in the field of optimum design of structures. Appl. Mech. Revs., Vol. 16, № 5, 1963.
3. Reitman, M. I. and Shapiro, G. S., The theory of optimal design in structural mechanics, theory of elasticity and plasticity. In: Results of Science. Elasticity and Plasticity. VINITI, Akad. Nauk SSSR, 1966.
4. Sheu, C. Y. and Prager, W., Recent developments in optimal structural design. Appl. Mech. Revs., Vol. 21, № 10, 1968.

Translated by E. D.

UDC 539.3

AXISYMMETRIC CONTACT PROBLEM FOR AN ELASTIC INHOMOGENEOUS HALF-SPACE IN THE PRESENCE OF COHESION

PMM Vol. 37, №6, 1973, pp. 1109-1116

G. Ia. POPOV
(Odessa)

(Received September 11, 1972)

There is obtained the exact solution of the axisymmetric contact problem on the indentation of a circular punch into an elastic half-space having a variable modulus of elasticity $E = E_\nu z^\nu$ $(0 \leqslant \nu < 1)$ in the case of the presence of complete cohesion.

1. For the formulation of the axisymmetric contact problem on the indentation of a circular punch into any linearly-deformable foundation, obviously, it is sufficient to know the vertical and radial displacements of the surface points of the foundation due to the action of vertical and radial loads of the form

$$p_0(r) = \delta(r - \rho), \qquad q_0(r) = \delta(r - \rho) \qquad (r, \rho \geqslant 0) \tag{1.1}$$

where $\delta(x)$ is Dirac's impulse function, describing in this case a concentrated load along a circumference (of radius ρ).

We adopt the following rule for the signs of the loads and displacements. The vertical loads and the corresponding displacements are considered to be positive if they are oriented downwards while the radial load and displacement is positive if they are orien-